Systems of higher-order initial-value problems

1. Convert this system of two 2nd-order IVPs into a system of four first-order IVPs

$$
\begin{aligned}
x^{(2)}(t) &= \sin(t) - 2y^{(1)}(t) - 3x(t) \\
x(0) &= 6 \\
x^{(1)}(0) &= 7 \\
y^{(2)}(t) &= \cos(t) - 4x^{(1)}(t) - 5y(t) \\
y(0) &= 8 \\
y^{(1)}(0) &= 9
\end{aligned}
$$

$$\mathbf{w}(t) = \begin{pmatrix} w_0(t) \\ w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ x^{(1)}(t) \\ y(t) \\ y^{(1)}(t) \end{pmatrix}$$

Answer: $\mathbf{w}^{(1)}(t) = \begin{pmatrix} w_1(t) \\ \sin(t) - 2w_3(t) - 3w_0(t) \\ w_3(t) \\ \cos(t) - 4w_1(t) - 5w_2(t) \end{pmatrix}$

$$\mathbf{w}(0) = \begin{pmatrix} 6 \\ 7 \\ 8 \\ 9 \end{pmatrix}$$

2. Convert this system of two IVPs into a system of three first-order IVPs

$$u^{(1)}(t) = -4tu(t)v(t)$$
$$u(0) = -3$$
$$v^{(2)}(t) = -7tu(t) + v(t) - 6v^{(1)}(t)$$
$$v(0) = 5$$
$$v^{(1)}(0) = 2$$

$$\mathbf{w}(t) = \begin{pmatrix} w_0(t) \\ w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ v(t) \\ v^{(1)}(t) \end{pmatrix}$$

Answer: $\mathbf{w}^{(1)}(t) = \begin{pmatrix} -4tw_0(t)w_1(t) \\ w_2(t) \\ -7tw_0(t) + w_1(t) - 6w_2(t) \end{pmatrix}$

$$\mathbf{w}(0) = \begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix}$$

3. Convert this system of two 3rd-order IVPs into a system of six first-order IVPs

$$x^{(3)}(t) = t - 5x(t)\,y^{(2)}(t) + y^{(1)}(t)$$
$$x(0) = 6$$
$$x^{(1)}(0) = 7$$
$$x^{(2)}(0) = -8$$
$$y^{(3)}(t) = \cos(t) - 4y(t)\,x^{(2)}(t) - x^{(1)}(t)$$
$$y(0) = 8$$
$$y^{(1)}(0) = 9$$
$$y^{(2)}(0) = -3$$

Answer:

$$\mathbf{w}(t) = \begin{pmatrix} w_0(t) \\ w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \\ w_5(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ x^{(1)}(t) \\ x^{(2)}(t) \\ y(t) \\ y^{(1)}(t) \\ y^{(2)}(t) \end{pmatrix}$$

$$\mathbf{w}^{(1)}(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \\ t - 5w_0(t)\,w_5(t) + w_4(t) \\ w_4(t) \\ w_5(t) \\ \cos(t) - 4w_3(t)\,w_2(t) - w_1(t) \end{pmatrix}$$

$$\mathbf{w}(0) = \begin{pmatrix} 6 \\ 7 \\ -8 \\ 8 \\ 9 \\ -3 \end{pmatrix}$$

4. How would you express the right-hand side of the ODE in Question 3 to a function in Matlab?

Answer:

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>> f = @(t, w)( [w(2); w(3); t - 5*w(1)*w(6) + w(5); w(5); w(6); ...
                                          cos(t) - 4*w(4)*w(3) - w(2)] );
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You must remember that in Matlab, vectors start at index 1, but for higher-order odes, it is often easier to understand if you represent the k^{th} derivative of the function by $w_k(t)$.

5. For Question 3, if you were to approximate $\mathbf{w}(t_k)$ with $\mathbf{w}_k$ where $t_k = t_0 + kh$, at each step, you would have a 6-dimensional vector. What do the entries of that 6-dimensional vector represent?

Answer: The first entry of the approximation $\mathbf{w}_k$ would approximate $x(t_k)$.

The second entry of the approximation $\mathbf{w}_k$ would approximate $x^{(1)}(t_k)$.

The third entry of the approximation $\mathbf{w}_k$ would approximate $x^{(2)}(t_k)$.

The fourth entry of the approximation $\mathbf{w}_k$ would approximate $y(t_k)$.

The fifth entry of the approximation $\mathbf{w}_k$ would approximate $y^{(1)}(t_k)$.

The sixth entry of the approximation $\mathbf{w}_k$ would approximate $y^{(2)}(t_k)$.

6. Suppose you had approximations to the solution as described in Question 6. Suppose you also wanted to approximate the solution on the interval $[t_k, t_{k+1}]$. Could you find two polynomials that matches the approximations of the function at t_k and t_{k+1} and also matches all three derivatives at each of the points?

Answer: In theory, yes, but you would have to use two splines, each of degree 7. There are eight constraints that are being described, and therefore you need a polynomial with eight unknowns; that is, a polynomial of degree seven. Given the approximation already has an error, a cubic spline is probably sufficient.